A Note on G- Optimal Stopping Problems

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Abstract

We consider a class of discretionary stopping problems within the G-framework. We first establish the well-definedness of the stopping problem under the G-expectation, by showing the quasi-continuity of the stopped process. We then prove a verification theorem for G-optimal stopping problem. One corollary is a direct proof for the well-known fact that the G-optimal stopping problem is the same as the classical optimal stopping problem with appropriate parameters, when the payoff function is concave or convex.

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1 Introduction

In this paper, we consider the following class of optimal stopping problem

$$v(x) = \sup_{\tau \in \mathcal{T}} \hat{\mathbb{E}}[e^{-r\tau} g(X_\tau)], \tag{1}$$

where $\hat{\mathbb{E}}$ is a G-expecation, g is a nonnegative and continuous function, $r > 0$ is a constant, $\mathcal{T}$ is the collection of stopping times τ which are continuous on $[\tau < \infty]$, and X is a G-Itô diffusion processes given by the following G-SDE

$$\begin{aligned} dX_t &= \mu(X_t)dt + \beta(X_t)d\langle B\rangle_t + \sigma(X_t)dB_t,\ t \geq 0, \\ X_0 &= x \in \mathbb{R}, \end{aligned} \tag{2}$$

with $\phi = \mu, \beta, \sigma : \mathbb{R} \to \mathbb{R}$ satisfying the following conditions: for $x, y \in \mathbb{R}$, there exists a constant $L > 0$ such that

(I) (linear growth) $|\phi(x)| \leq L(1 + |x|)$,

(II) (Lipschitz continuity) $|\phi(x) - \phi(y)| \leq L|x - y|$.

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For the existence and uniqueness of the solution of (2), we refer to Gao [G09]. Although the condition (I) here is just replaced by a boundedness condition there, the proof is similar except some technical details, which is given in the appendix.

Here $B_{.}$ is a one dimensional G-Brownian motion on a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ with $B_1 = N(\{0\} \times [\underline{\sigma}^2, \overline{\sigma}^2])$ and $\mathbb{F} = \{\mathcal{F}_t^B\}$ be the natural filtration of $(B_t)_{t\geq 0}$. We also adopt the tradition that on $[\tau = \infty]$, $e^{-r\tau} g(X_\tau) = 0$.

We first establish the well-definedness of the stopping problem under the G-expectation, by showing the quasi-continuity of the stopped process. We then prove a verification theorem for G-optimal stopping problem. One corollary is a direct proof for the well-known fact that the G-optimal stopping problem is the same as the classical optimal stopping problem with appropriate parameters, when the payoff function is concave or convex.

2 Preliminary

We first recall some basics of G-expectation, G-Browian motion and stopping times under G-framework.

2.1 G-expectation and G-Brownian Motion

Let Ω be a given set and let $\mathcal{H}$ be a linear space of real valued functions on Ω. And we suppose that $\mathcal{H}$ satisfies $c \in \mathcal{H}$,for all $c \in \mathbb{R}$, and $|X| \in \mathcal{H}$, when $X \in \mathcal{H}$. We consider such space as the space of random variables.

Definition 1 (Sublinear expectation) *A sublinear expectation $\mathbb{E}$ is a functional on $\mathcal{H}$ satisfying the following conditions*
(i) *Monotonicity: $\mathbb{E}[X] \geq \mathbb{E}[Y]$, if $X \geq Y$.*
(ii) *Constant preserving: $\mathbb{E}[c] = c$, for $c \in \mathbb{R}$.*
(iii) *Sub-additivity: $\mathbb{E}[X+Y] \leq \mathbb{E}[X] + \mathbb{E}[Y]$.*
(iv) *Positive homogeneity: $\mathbb{E}[aX] = a\mathbb{E}[X]$, for $a \geq 0$.*
The triple $(\Omega, \mathcal{H}, \mathbb{E})$ is called a sublinear expectation space.

Definition 2 *Let X_1 and X_2 be two random variables defined respectively in sublinear expectation spaces $(\Omega_1, \mathcal{H}_1, \mathbb{E}_1)$ and $(\Omega_2, \mathcal{H}_2, \mathbb{E}_2)$. They are called identically distributed, denoted by $X_1 \overset{d}{=} X_2$, if $\mathbb{E}_1[\varphi(X_1)] = \mathbb{E}_2[\varphi(X_2)], \forall \varphi \in C_{l.Lip}(\mathbb{R})$, where $C_{l.Lip}(\mathbb{R})$ is the space of real valued continuous functions defined on $\mathbb{R}$ such that*

$$|\varphi(x) - \varphi(y)| \leq C(1 + |x|^k + |y|^k)|x-y|, \forall\, x, y \in \mathbb{R},$$

for some $C > 0, k \in \mathbb{N}$ depending on φ.

Definition 3 (Independence) *In a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$ a random variable Y is said to be independent to another random variable X under $\mathbb{E}$ if for each test function $\varphi \in C_{l.Lip}(\mathbb{R}^2)$ we have*

$$\mathbb{E}[\varphi(X, Y)] = \mathbb{E}[\mathbb{E}[\varphi(x, Y)]_{x=X}].$$

Definition 4 (G-normal distribution) *A random variable X in a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$ is called G-normal distributed if*

$$aX + b\bar{X} \overset{d}{=} \sqrt{a^2 + b^2}X \quad \textit{for}\, a, b \geq 0$$

where $\bar{X}$ is an independent copy of X. Here the letter G denotes the function

$$G(\gamma) = \frac{1}{2}\mathbb{E}[\gamma X^2] : \mathbb{R} \to \mathbb{R}.$$

The function $G(\cdot) : \mathbb{R} \to \mathbb{R}$ is a monotonic, sublinear function, and has a representation

$$G(\gamma) = \frac{1}{2} \sup_{\alpha \in [\underline{\sigma}^2, \overline{\sigma}^2]} \gamma\alpha,$$

where $\underline{\sigma}^2 = -\mathbb{E}[-X^2]$ and $\overline{\sigma}^2 = \mathbb{E}[X^2]$. In this paper, we always assume that $\underline{\sigma}^2 > 0$. We denote the G-normal distribution by $N(0 \times [\underline{\sigma}^2, \overline{\sigma}^2])$.

Remark 1 *We can rewrite G as $G(\gamma) = \frac{1}{2}(\overline{\sigma}^2\gamma^+ - \underline{\sigma}^2\gamma^-)$. In particular, when γ is positive, $G(\gamma) = \frac{1}{2}\overline{\sigma}^2\gamma^+$, and when γ is negative, $G(\gamma) = \frac{1}{2}\underline{\sigma}^2\gamma^-$.*

Definition 5 *A stochastic process X defined in a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$ is a family of random variables X_t parametrized by $t \in [0, \infty)$ such that $X_t \in \mathcal{H}$ for each t.*

Now we give the definition of G-Brownian motion.

Definition 6 *Let $G(\cdot)$ be given as before. A process $(B_t)_{t\geq 0}$ on a sublinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$ is called a G-Brownian motion if the following properties are satisfied*
(i) $B_0(\omega) = 0, \forall\omega \in \Omega,$
(ii) *for each $t, s \geq 0$, the increment $B_{t+s} - B_t$ is $N(\{0\} \times s[\underline{\sigma}^2, \overline{\sigma}^2])$-distributed and is independent from $(B_{t_1}, B_{t_2}, \cdots, B_{t_n})$ for each $n \in \mathbb{N}$ and $0 \leq t_1 \leq \cdots \leq t_n \leq t$.*

Now we specify the sublinear expectation space. We let $\Omega = C_0[0, \infty)$ be the space of all real valued continuous functions $(\omega_t)_{t\geq 0}$ with $\omega_0 = 0$, equipped with the metric

$$\rho(\omega_1, \omega_2) := \sum_{n=1}^{\infty} \frac{1}{2^n} \max_{0\leq t\leq n}(|\omega_1(t) - \omega_2(t)| \wedge 1),\ \omega_1, \omega_2 \in \Omega.$$

For each $T \in [0, \infty)$, we set $\Omega_T := \{\omega(\cdot \wedge t) : \omega \in \Omega\}$. We will consider the canonical process $B_t(\omega) = \omega(t), t \in [0, \infty), \text{for}\, \omega \in \Omega$.

For each fixed $T \in [0, \infty)$, we set

$$L_{ip}(\Omega_T) := \{\varphi(B_{t_1\wedge T}, \cdots, B_{t_n\wedge T}) : n \in \mathbb{N}, t_1, \cdots, t_n \in [0, \infty), \varphi \in C_{l.Lip}(\mathbb{R}^n)\}.$$

Let $\mathcal{H} \equiv L_{ip}(\Omega) := \bigcup_{n=1}^{\infty} L_{ip}(\Omega_n)$.

Then one can construct a sublinear expectation $\hat{\mathbb{E}}$ on $(\Omega, \mathcal{H})$ as Peng [P10] does, such that the canonical process is a G-Brownian motion on the sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$.

The related conditional G-expectation of $X = \varphi(B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_{t_n} - B_{t_{n-1}})$ under Ω_{t_j} is defined by

$$\hat{\mathbb{E}}_{t_j}[X] = \hat{\mathbb{E}}[\varphi(x_1, \cdots, x_j, B_{t_{j+1}} - B_{t_j}, \cdots, B_{t_n} - B_{t_{n-1}})]_{(x_1,\cdots,x_j)=(B_{t_1},\cdots,B_{t_j}-B_{t_{j-1}})}.$$

Remark 2 *From the definition of conditional G-expectation, one can see that* $\hat{\mathbb{E}}_t[\xi] = \xi$, *for each* $\xi \in L_{ip}(\Omega_t)$.

We denote by $L_G^p(\Omega), p \geq 1$, the completion of $L_{ip}(\Omega)$ under the norm $\|X\|_p := (\hat{\mathbb{E}}[|X|^p])^{1/p}$. Similarly, we can define $L_G^p(\Omega_t)$. We can continuously extend $\hat{\mathbb{E}}[\cdot]$ to a sublinear expectation on $(\Omega, L_G^1(\Omega))$ still denoted by $\hat{\mathbb{E}}$. Similarly, we can continuously extend the conditional G-expectation $\hat{\mathbb{E}}_t[\cdot]$ to be $\hat{\mathbb{E}}_t[\cdot] : L_G^1(\Omega) \to L_G^1(\Omega_t)$.

Definition 7 (G-martingale) *A process* $(M_t)_{t\geq 0}$ *is called a G-martingale (respectively, G-supermartingale, G-submartingale) if for each* $t \in [0, \infty), M_t \in L_G^1(\Omega_t)$ *and for each* $s \in [0, t]$, *we have*

$$\hat{\mathbb{E}}_s[M_t] = M_s \quad (\textit{respectively}, \leq M_s, \geq M_s).$$

2.2 G-Itô calculus

Now we recall the definition of G-Itô's integral.

Let $p \geq 1$ be fixed. We consider the following type of simple processes: for a given partition $\pi_T = \{t_0, \cdots, t_N\}$ of $[0, T]$ we set

$$\eta_t = \sum_{k=0}^{N-1} \xi_k(\omega) \mathbf{1}_{[t_k, t_{k+1})}(t),$$

where $\xi_k \in L_G^p(\Omega_{t_k}), k = 0, 1, \cdots, N-1$ are given. The collection of these processes is denoted by $M_G^{p,0}(0, T)$.

Definition 8 *For an* $\eta \in M_G^{p,0}(0, T)$ *with* $\eta_t = \sum_{k=0}^{N-1} \xi_k(\omega) \mathbf{1}_{[t_k, t_{k+1})}(t)$, *the related Bochner integral is*

$$\int_0^T \eta_t(\omega)\, dt := \sum_{k=0}^{N-1} \xi_k(\omega)(t_{t+1} - t_k).$$

We denote by $M_G^p(0, T)$ the completion of $M_G^{p,0}(0, T)$ under the norm

$$\|\eta\|_{M_G^p(0,T)} = \left\{ \hat{\mathbb{E}}[\int_0^T |\eta_t|^p\, dt] \right\}^{1/p}.$$

Definition 9 (G-Itô's integral) *For each* $\eta \in M_G^{2,0}(0, T)$ *of the form*

$$\eta_t = \sum_{k=0}^{N-1} \xi_k(\omega) \mathbf{1}_{[t_k, t_{k+1})}(t)$$

we define

$$I(\eta) = \int_0^T \eta_t\, dB_t := \sum_{k=0}^{N-1} \xi_k (B_{t_{k+1}} - B_{t_k}).$$

Lemma 1 *The mapping $I : M_G^{2,0}(0,T) \to L_G^2(\Omega_T)$ is a continuous linear mapping and thus can be continuously extended to $I : M_G^2(0,T) \to L_G^2(\Omega_T)$. And we have*

$$\hat{\mathbb{E}}[\int_0^T \eta_t \, dB_t] = 0,$$
$$\hat{\mathbb{E}}[(\int_0^T \eta_t \, dB_t)^2] \le \overline{\sigma}^2 \hat{\mathbb{E}}[\int_0^T \eta_t^2 \, dt].$$

Let $\pi_t^N, N = 1, 2, \cdots$, be a sequence of partitions of $[0,t]$. We consider

$$\begin{aligned} B_t^2 &= \sum_{j=0}^{N-1} (B_{t_{j+1}^N}^2 - B_{t_j^N}^2) \\ &= \sum_{j=0}^{N-1} 2B_{t_j^N}(B_{t_{j+1}^N} - B_{t_j^N}) + \sum_{j=0}^{N-1} (B_{t_{j+1}^N} - B_{t_j^N})^2. \end{aligned}$$

As $\mu(\pi_t^N) = \max\{|t_{j+1} - t_j| : j = 0, 1, \cdots, N-1\} \to 0$, the first term of the right side converges to $2\int_0^t B_s \, dB_s$ in $L_G^2(\Omega)$. We denote the quadratic variation process of G-Brownian motion by $\langle B \rangle_t$, i.e.,

$$\langle B \rangle_t = \lim_{\mu(\pi_t^N) \to 0} \sum_{j=0}^{N-1} (B_{t_{j+1}^N} - B_{t_j^N})^2 = B_t^2 - 2\int_0^t B_s \, dB_s \quad \text{in } L_G^2(\Omega).$$

By the above construction, $(\langle B \rangle_t)_{t \ge 0}$ is an increasing process with $\langle B \rangle_0 = 0$.

Similar to G-Itô's integral with respect to G-Brownian motion B, we can define the integral of a process $\eta \in M_G^1(0,T)$ with respect to $\langle B \rangle$: $\int_0^T \eta_t \, d\langle B \rangle_t$.

Lemma 2 *Let $\eta \in M_G^2(0,T)$. Then*

$$\hat{\mathbb{E}}[(\int_0^T \eta_t \, dB_t)^2] = \hat{\mathbb{E}}[\int_0^T \eta_t^2 \, d\langle B \rangle_t].$$

Lemma 3 *Let $\eta \in M_G^1(0,T)$. Then*

$$\underline{\sigma}^2 \hat{\mathbb{E}}[\int_0^T |\eta_t| \, dt] \le \hat{\mathbb{E}}[\int_0^T |\eta_t| \, d\langle B \rangle_t] \le \overline{\sigma}^2 \hat{\mathbb{E}}[\int_0^T |\eta_t| \, dt].$$

Now we list G-Itô's formula here.

Proposition 4 (G-Itô's formula) *Let $\Phi \in C^{1,2}([0,T] \times \mathbb{R})$ and*

$$X_t = X_0 + \int_0^t \alpha_s \, ds + \int_0^t \eta_s \, d\langle B \rangle_s + \int_0^t \beta_s \, dB_s,$$

where $\alpha, \eta \in M_G^1(0,T), \beta \in M_G^2(0,T)$. Then for each $t \in [0,T]$, we have

$$\begin{aligned} \Phi(t, X_t) - \Phi(0, X_0) = &\int_0^t \partial_x \Phi(u, X_u) \beta_u \, dB_u + \int_0^t [\partial_t \Phi(u, X_u) + \partial_x \Phi(u, X_u) \alpha_u] \, du \\ &+ \int_0^t \left[\partial_x \Phi(u, X_u) \eta_u + \frac{1}{2} \partial_{xx}^2 \Phi(u, X_u) \beta_u^2 \right] d\langle B \rangle_u. \end{aligned}$$

We denote by $\bar{M}_G^p(0,T)$, $p \geq 1$, the completion of $M_G^{p,0}(0,T)$ under the norm $\|\eta\|_{\bar{M}_G^p(0,T)} = (\int_0^T \hat{\mathbb{E}}[|\eta_t|^p]\, dt)^{1/p}$. It is easy to see $\bar{M}_G^p(0,T) \subset M_G^p(0,T)$.

We consider the following SDE driven by G-Brownian motion

$$X_t = X_0 + \int_0^t \mu(X_u)\, du + \int_0^t \beta(X_u)\, d\langle B\rangle_u + \int_0^t \sigma(X_u)\, dB_u,\ t \geq 0 \tag{3}$$

where $X_0 \in \mathbb{R}$ is a given constant, and $\phi = \mu, \beta, \sigma : \mathbb{R} \to \mathbb{R}$ satisfying the following conditions: for $x, y \in \mathbb{R}$, there exists a constant $L > 0$ such that

(I) (linear growth) $|\phi(x)| \leq L(1 + |x|)$,

(II) (Lipschitz continuity) $|\phi(x) - \phi(y)| \leq L|x - y|$.

We can mimic the classical situation to get the existence and the uniqueness of the solution of the above SDE driven by G-Brownian motion. We refer to Gao [G09].

Proposition 5 (Gao [G09]) *Let (I)(II) be hold. Then there exists a unique continuous process $X = \{X_t, t \geq 0\}$ which satisfies SDE (3) such that for any $t \geq 0, X_t \in L_G^1(\Omega_t)$, and for each $p \geq 2$ and fixed $T > 0, X\mathbf{1}_{[0,T]} \in \bar{M}_G^p(0,T)$.*

2.3 Quasi-surely Analysis under G-capacity

Let $\mathcal{B}(\Omega)$ be the Borel σ-algebra on Ω. We now give the following notations

- $L^0(\Omega)$: the space of all $\mathcal{B}(\Omega)$-measurable real functions;
- $B_b(\Omega)$: all the bounded functions in $L^0(\Omega)$;

We give the representation of G-expectation.

Proposition 6 (Denis, Hu, Peng [DHP11]) *There exists a weakly compact family of probability measures $\mathcal{P}$ on $(\Omega, \mathcal{B}(\Omega))$ such that*

$$\hat{\mathbb{E}}[X] = \max_{P \in \mathcal{P}} E^P[X],\ \textit{for}\ X \in L_{ip}(\Omega),$$

where E^P is the linear expectation with respect to P.

Remark 3 *In fact, we can construct the family $\mathcal{P}$ in a more explicit way. Let $(\Omega, \mathcal{F}, P^0)$ be a filtered probability space, and $\{W_t\}_{t\geq 0}$ be a classical standard Brownian motion under P^0. Then we have the representation*

$$\hat{\mathbb{E}}[X] = \sup_{P \in \mathcal{P}_M} E^P[X], X \in L_{ip}(\Omega), \tag{4}$$

where $\mathcal{P}_M$ is the weak closure of $\mathcal{P}'_M$, with

$$\mathcal{P}'_M := \{P^0 \circ (B^\gamma)^{-1} : B_t^\gamma = \int_0^t \gamma_s dW_s, \gamma \in L_{\mathcal{F}}^2([0,T]; [\underline{\sigma}, \overline{\sigma}])\}.$$

Moreover, under each $P \in \mathcal{P}_M$, B is a martingale.

Definition 10 (G-capacity) *For the $\mathcal{P}$ in the representation of G-expectation, we can define the associated G-capacity:*

$$c(A) := \sup_{P\in\mathcal{P}} P(A), \quad A \in \mathcal{B}(\Omega)$$

and the upper expectation $\bar{\mathbb{E}}$ which makes the following definition meaningful

$$\bar{\mathbb{E}}[X] := \sup_{P\in\mathcal{P}} E^P[X], \text{ for } X \in L^0(\Omega).$$

Definition 11 *A set $A \in \mathcal{B}(\Omega)$ is called a polar if $c(A) = 0$ and a property holds 'quasi-surely' (q.s.) if it holds outside a polar set.*

Definition 12 (1) *A mapping X on Ω with values in a topological space is said to be quasi-continuous(q.c.) if for any $\varepsilon > 0$, there exists an open set $O \in \mathcal{B}(\Omega)$ with $c(O) < \varepsilon$ such that $X|_{O^c}$ is continuous.*
(2) *We say that $X : \Omega \to \mathbb{R}$ has a quasi-continuous version if there exists a quasi-continuous function $Y : \Omega \to \mathbb{R}$ with $X = Y$ q.s..*

We set, for $p > 0$,

- $\mathcal{L}^p := \{X \in L^0(\Omega) : \bar{\mathbb{E}}[|X|^p] < \infty\}$;
- $\mathcal{N}^p := \{X \in L^0(\Omega) : \bar{\mathbb{E}}[|X|^p] = 0\}$;
- $\mathcal{N} := \{X \in L^0(\Omega) : X = 0, \text{ q.s.}\}$.

Noticing that $\mathcal{N}^p = \mathcal{N}$, we denote $L^p := \mathcal{L}^p/\mathcal{N}$. When $p \geq 1$, we see that L^p is a Banach space under the norm $\|X\|_p := (\bar{\mathbb{E}}[|X|^p])^{1/p}$.

For each $p \geq 1$, let $L_b^p(\Omega)$ be the completion of $\mathcal{B}_b(\Omega)$ under $(\bar{E}[|\cdot|^p])^{1/p}$ respectively. And Denis, Hu and Peng [DHP11] have the following result.

Proposition 7 *For each $p \geq 1$,*

$$L_G^p(\Omega) = \{X \in L^0(\Omega) : X \text{ has a quasi-continuous version and } \lim_{n\to\infty} \bar{\mathbb{E}}[|X|^p \mathbf{1}_{[|X|>n]}] = 0\}.$$

Furthermore, $\bar{\mathbb{E}}[X] = \hat{\mathbb{E}}[X]$, for each $X \in L_G^1(\Omega)$.

Proposition 8 (Monotone convergence theorem) *1. Let $X_n \in L_G^1(\Omega)$. If $X_n \downarrow X$, q.s., then $\hat{\mathbb{E}}[X_n] \downarrow \bar{\mathbb{E}}[X]$.*

2. Let $X_n \in L^0(\Omega)$ and lower bounded (in particular, nonnegative). If $X_n \uparrow X$, q.s., then $\bar{\mathbb{E}}[X_n] \uparrow \bar{\mathbb{E}}[X]$.

Remark 4 *Note that $\bar{\mathbb{E}} = \hat{\mathbb{E}}$ on $L_{ip}(\Omega)$ by the Proposition 6, we know that the $(\hat{\mathbb{E}}[|\cdot|^p])^{1/p}$-completion and the $(\bar{E}[|\cdot|^p])^{1/p}$-completion of $L_{ip}(\Omega)$ are the same, which is $L_G^p(\Omega)$. Therefore, $L_G^p(\Omega)$ is a closed subspace of L^p, which implies that for any $X \in L^p$ if there exists a sequence of random variables $X_n \in L_G^p(\Omega)$ such that $X_n \to X$ in L^p, then $X \in L_G^p(\Omega)$.*

Lemma 9 (Fatou's lemma) *For any sequence of nonnegative random variables $X_n \in L^0(\Omega)$, we have $\bar{\mathbb{E}}[\liminf_n X_n] \leq \liminf_n \bar{\mathbb{E}}[X_n]$.*

Definition 13 (Uniform integrability) *Consider $D \subset L^1$. D is said to be uniformly integrable (U.I.) if $\bar{\mathbb{E}}[|X|\mathbf{1}_{[|X|>n]}]$ converges to 0 uniformly in $X \in D$ as $n \to \infty$.*

Proposition 10 (Equivalent conditions of U.I.) *Suppose D is a subset of L^1. Then D is U.I. if and only if both*
(1) $\sup_{X\in D} \bar{\mathbb{E}}[|X|] < \infty$;
(2) *for any $\varepsilon > 0$ there is a $\delta > 0$ s.t. for $\forall A \in \mathcal{F}$ with $\bar{\mathbb{E}}[I_A] \leq \delta$ we have $\bar{\mathbb{E}}[|X|I_A] < \varepsilon$ for $\forall X \in D$.*

Corollary 11 *Let D be a subset of L^1. Suppose there is a positive function ϕ defined on $[0, \infty)$ s.t. $\lim_{t\to\infty} t^{-1}\phi(t) = \infty$ and $\sup_{X\in D} \bar{\mathbb{E}}[\phi \circ |X|] < \infty$. Then D is uniformly integrable.*

Remark 5 *A special case is $\phi(t) = t^q, t \in [0, \infty), q > 1$.*

Definition 14 (Convergence in capacity) *A sequence $X_n \in L^1$ is said to converge in capacity to some X_∞, if for any $\varepsilon, \delta > 0$, there exists an $N \in \mathbb{N}$ s.t.*

$$\bar{\mathbb{E}}[I_{\{|X_m - X_\infty|>\varepsilon\}}] < \delta, \quad \text{for } \forall m \geq N.$$

Remark 6 *A sequence $\{X_n\}_{n\in\mathbb{N}}$ converging q.s. is a sequence converging in capacity.*

Proposition 12 *Suppose X_n is a sequence in L^1_b, and $X \in L^1$. Then X_n converge in L^1 norm to X iff the collection $\{X_n\}_{n\in\mathbb{N}}$ is uniformly integrable and the X_n converge in capacity to X.*

Furthermore, in this case, the collection $\{X_n\}_{n\in\mathbb{N}} \cup \{X\}$ is also uniformly integrable and $X \in L^1_b$.

3 Well-posedness of Problem (1)

Given all the analytical preliminary, it is clear that G-optimal stopping problem is not *a priori* well defined under any sublinear space for any process and payoff functions.

We will establish the well-posedness in several steps. We first establish technical lemmas that will be useful for subsequent analysis.

3.1 Technical Lemmas

Lemma 13 *For any $a \in \mathbb{R}$, all $\delta > 0$ and $t \geq 0$, we have*

$$\hat{\mathbb{E}}[\int_0^t \mathbf{1}_{[a,a+\delta]}(B_s)d\langle B\rangle_s] \leq C\delta.$$

Here C is a constant which depends only on t.

Proof For $\delta > 0$, we define the C^2-function $h : \mathbb{R} \to \mathbb{R}$ such that $h(0) = 0, |h'(x)| \le \delta^{-1}, x \in (-\infty, a-\delta]$, and

$$h''(x) = \begin{cases} 0, & \text{if } x \le a-\delta, \\ \dfrac{x-a+\delta}{\delta^3}, & \text{if } a-\delta < x \le a, \\ \dfrac{1}{\delta^2}, & \text{if } a < x \le a+\delta, \\ -\dfrac{x-a-2\delta}{\delta^3}, & \text{if } a+\delta < x \le a+2\delta, \\ 0, & \text{if } x \ge a+2\delta. \end{cases}$$

Then we have $|h'| \le 4\delta^{-1}$ and $|h''| \le \delta^{-2}$.

By applying the G-Itô's formula to $h(B_t)$, we have

$$h(B_t) = \int_0^t h'(B_s)dB_s + \frac{1}{2}\int_0^t h''(B_t)d\langle B\rangle_s.$$

Therefore,

$$\begin{aligned} \int_0^t \mathbf{1}_{[a,a+\delta]}(B_s)d\langle B\rangle_s &\le \delta^2 \int_0^t h''(B_s)d\langle B\rangle_s \\ &= 2\delta^2 h(B_t) - 2\delta^2 \int_0^t h'(B_s)dB_s \\ &\le 8\delta|B_t| - 2\delta^2 \int_0^t h'(B_s)dB_s. \end{aligned}$$

It follows that

$$\begin{aligned} \hat{\mathbb{E}}[\int_0^t \mathbf{1}_{[a,a+\delta]}(B_s)d\langle B\rangle_s] &\le \hat{\mathbb{E}}[8\delta|B_t| - 2\delta^2 \int_0^t h'(B_s)dB_s] \\ &= 8\delta\hat{\mathbb{E}}[|B_t|] = C\delta. \end{aligned}$$

□

Lemma 14 (G-Ito's for smooth function) *Suppose that $f(x)$ is C^2 except on a finite number of points $a_1, a_2, \cdots, a_n$ where $f(x)$ is C^1. Furthermore, we assume that f'' has finite jumps at these points. Then we have*

$$f(B_t) = f(0) + \int_0^t f'(B_s)dB_s + \frac{1}{2}\int_0^t f''(B_s)d\langle B\rangle_s \tag{5}$$

Proof It suffices to show for $n = 1$ with $a_1 = a$.

First we define $f''(a)$ such that f'' is semicontinuous, e.g. if $f''(a+) > f''(a-)$ and we set $f''(a) := f''(a-)$, then f'' is lower semicontinuous (LSC).

WLOG, we assume that $f''(a+) > f''(a-)$. On the one hand, we set $f''(a) := f''(a-)$, and f'' is LSC. Then there exists $\{\underline{\rho}''_n\} \subset C(\mathbb{R})$, such that $\underline{\rho}''_n \le f''$, increases to f'' and satisfies $\underline{\rho}''_n = f''$ on $\mathbb{R} \setminus [a - \frac{1}{n}, a + \frac{1}{n}]$.

Now we set

$$\underline{\rho}_n'(x) = \begin{cases} f'(x), & x < a-1, \\ \int_{a-1}^{x} \underline{\rho}_n''(y)\, dy + f'(a-1), & x \geq a-1, \end{cases}$$

and

$$\underline{\rho}_n(x) = \begin{cases} f(x), & x < a-1, \\ \int_{a-1}^{x} \underline{\rho}_n'(y)\, dy + f(a-1), & x \geq a-1. \end{cases}$$

We observe that $\underline{\rho}_n$ is of class C^2 and increases to f, $\underline{\rho}_n'$ increases to f', and $\underline{\rho}_n''$ increases to f'' pointwise.

Now applying G-Itô's formula to $\underline{\rho}_n$, we obtain that

$$\underline{\rho}_n(B_t) = \underline{\rho}_n(0) + \int_0^t \underline{\rho}_n'(B_s)dB_s + \frac{1}{2}\int_0^t \underline{\rho}_n''(B_s)d\langle B\rangle_s.$$

We have

$$\underline{\rho}_n(B_t) \to f(B_t), \quad \underline{\rho}_n(0) \to f(0).$$

And by G-Itô isometry and monotone convergence, we obtain

$$\int_0^t \underline{\rho}_n'(B_s)dB_s \to \int_0^t f'(B_s)dB_s$$

in L_G^2, as $n \to \infty$.

By Remark 4, noticing that $\underline{\rho}_n'' = f''$ on $(-\infty, a-\frac{1}{n}) \cup (a+\frac{1}{n}, \infty)$, and $\hat{\mathbb{E}}[|\underline{\rho}_n''(B_s) - f''(B_s)|]$ decreases to 0 (monotone convergence), we have

$$\begin{aligned}
&\hat{\mathbb{E}}[|\int_0^t \underline{\rho}_n''(B_s)d\langle B\rangle_s - \int_0^t f''(B_s)d\langle B\rangle_s|] \\
\leq &\hat{\mathbb{E}}[\int_0^t |\underline{\rho}_n''(B_s) - f''(B_s)|d\langle B\rangle_s] \\
\leq &\overline{\sigma}^2 \int_0^t \hat{\mathbb{E}}[|\underline{\rho}_n''(B_s) - f''(B_s)|\mathbf{1}_{[a-\frac{1}{n}, a+\frac{1}{n}]}(B_s)]\, ds \to 0, \text{as } n \to \infty.
\end{aligned}$$

On the other hand, if we set $f''(a) := f''(a+)$, then f'' is upper semicontinuous. Hence, there exists $\{\overline{\rho}_n''\} \subset C(\mathbb{R})$, such that $\overline{\rho}_n'' \geq f''$, decreases to f'' and satisfies $\overline{\rho}_n'' = f''$ on $\mathbb{R} \setminus [a-\frac{1}{n}, a+\frac{1}{n}]$. Similarly, we can define $\overline{\rho}_n'$ and $\overline{\rho}_n$ decreasing to f' and f respectively. Moreover, noticing that $0 \leq \overline{\rho}_n'' - \underline{\rho}_n'' \leq M$ for some positive constant M and $\overline{\rho}_n'' = \underline{\rho}''$ on $\mathbb{R} \setminus [a-\frac{1}{n}, a+\frac{1}{n}]$, we have

$$\begin{aligned}
&\hat{\mathbb{E}}[|\int_0^t \underline{\rho}_n''(B_s)d\langle B\rangle_s - \int_0^t \overline{\rho}''(B_s)d\langle B\rangle_s|] \\
\leq &\hat{\mathbb{E}}[\int_0^t |\underline{\rho}_n''(B_s) - \overline{\rho}''(B_s)|d\langle B\rangle_s] \\
\leq &M\hat{\mathbb{E}}[\int_0^t \mathbf{1}_{[a-\frac{1}{n}, a+\frac{1}{n}]}(B_s)\, d\langle B\rangle_s] \\
\leq &MC\frac{2}{n} \to 0, \text{as } n \to \infty,
\end{aligned}$$

where we use Lemma 13 in the last inequality. Now we have proved (5). ∎

As the above two lemmas, Lemma 13 and Lemma 14, we can prove the following lemma which we will use in the proof for the main theorem. Here we need the estimate of the process X, given in Remark 8.

Lemma 15 (a) *For any $a \in \mathbb{R}$, all $\delta > 0$ and $t \geq 0$, we have*

$$\hat{\mathbb{E}}[\int_0^t \mathbf{1}_{[a,a+\delta]}(X_s) d\langle B \rangle_s] \leq C\delta,$$

where C is a constant which depends only on t.
(b) *Suppose that f satisfies the conditions as in Lemma 14, then we have*

$$f(X_t) = f(X_0) + \int_0^t f'(X_s) dX_s + \frac{1}{2} \int_0^t f''(X_s) d\langle B \rangle_s. \tag{6}$$

Lemma 16 *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function continuous on $\mathbb{R} \setminus \{a\}$, and it has polynomial growth, i.e. there exist a positive constant C and an integer $m \in \mathbb{N}$ such that $|f(x)| \leq C(1 + |x|^m), \forall x \in \mathbb{R}$. Then we have $f(X)\mathbf{1}_{[0,T]} \in M_G^p(0,T)$.*

<u>Proof</u>. The proof is similar to the above one. ∎

Lemma 17 *Let $M_0 \in \mathbb{R}, \psi \in M_G^2(0,T)$ and $\eta \in M_G^1(0,T)$ be given and let*

$$M_t = M_0 + \int_0^t \psi_u \, dB_t + \int_0^t \eta_u \, d\langle B \rangle_u - 2 \int_0^t G(\eta_u) \, du \quad \text{for } t \in [0,T]. \tag{7}$$

Then M is a G-martingale. In particular,

$$\hat{\mathbb{E}}_s \left[\int_0^t \eta(u) d\langle B \rangle_u - 2 \int_0^t G(\eta(u)) du \right] = \int_0^s \eta(u) d\langle B \rangle_u - 2 \int_0^s G(\eta(u)) du, \; 0 \leq s < t.$$

This is from Prop 1.4 Chap. IV in Peng [P10].

3.2 Well-posedness

Now we can establish the well-posedness of Problem (1) via the following Propositions.

Proposition 18 *For each $T > 0$, the solution of Eqn.* (3) *$X = \{X_t(\omega); 0 \leq t \leq T, \omega \in \Omega\}$ has a quasi continuous version.*

<u>Proof</u>. From the proof of Proposition 5, we know that $X_t = \lim_{m \to \infty} X_t^{(m)}$ and

$$\bar{\mathbb{E}}[\sup_{0 \leq s \leq T} |X_s^{(m)} - X_s^{(m+l)}|^p] \to 0$$

for any $l \geq 1$ as $m \to \infty$. Therefore if for each $m \geq 0, X^{(m)} = \{X_t^{(m)}; 0 \leq t \leq T\}$ has a quasi continuous version, then we can choose them to be quasi continuous and use the same arguments as in Corollary 5.2 in Song [S11a] to get our result.

First, notice that $X^{(0)}$ is quasi continuous, and by the construction of Picard sequence, we have

$$X_t^{(1)} = X_0 + \mu(X_0)t + \beta(X_0)\langle B\rangle_t + \sigma(X_0)B_t,\ 0 \leq t \leq T,$$

which also has a quasi continuous version. Now we claim that, if $X^{(m)}$ has a quasi continuous version, then $X^{(m+1)}$ also has a quasi continuous version.

Let $\psi : \mathbb{R} \to \mathbb{R}$ satisfy conditions (I)(II), then $\psi(X^{(m)})$ has a quasi continuous version. Assume that $\psi(X^{(m)})$ is quasi continuous, i.e. for any $\varepsilon > 0$ there exists an open subset $O \subset \Omega$ with $c(O) < \varepsilon$ such that $\psi(X^{(m)})$ is continuous on $O^c \times [0,T]$. Set $\Psi(t) = \int_0^t \psi(X_u^{(m)})\,du$, then $\Psi(\cdot)$ is continuous on $O^c \times [0,T]$, thus quasi continuous.

Note that

$$\begin{aligned}
X_t^{(m+1)} =& X_0 + \int_0^t \mu(X_u^{(m)})\,du + \int_0^t \beta(X_u^{(m)})\,d\langle B\rangle_u + \int_0^t \sigma(X_u^{(m)})\,dB_u \\
=& X_0 + \int_0^t \mu(X_u^{(m)})\,du + \int_0^t G(\beta(X_u^{(m)}))\,du \\
&+ \int_0^t \beta(X_u^{(m)})\,d\langle B\rangle_u - \int_0^t G(\beta(X_u^{(m)}))\,du + \int_0^t \sigma(X_u^{(m)})\,dB_u,
\end{aligned}$$

where the summation of last three terms is a martingale by Lemma 17. We have shown the first three terms have quasi continuous versions, and by Theorem 5.3 in Song [S11a] we get the summation of last three terms has a quasi continuous version, therefore $X^{(m+1)}$ has a quasi continuous version. We are done. $\square$

Proposition 19 *Let τ be a stopping time. If for any $T > 0$, $\tau \wedge T$ has a q.c. version, then for fixed $0 \leq s < t \leq T, \mathbf{1}_{[s,t]}(\tau \wedge T)$ has a q.c. version, or $\mathbf{1}_{[s,t]}(\tau \wedge T) \in L_G^1(\Omega)$.*

Proof. Assume that $\tau \wedge T$ is q.c., then for $\forall \varepsilon > 0$, there exists an open set $O \subset \Omega$ open subset with $c(O) < \varepsilon$, such that $\tau \wedge T$ is continuous on O^c. Notice that $[\mathbf{1}_{[s,t]}(\tau \wedge T) < \delta]$ is an open subset in O^c for $\delta \in \mathbb{R}$, $\mathbf{1}_{[s,t]}(\tau \wedge T)$ is USC. Then there exist ζ_n bounded and continuous on O^c, and $\zeta_n \downarrow \mathbf{1}_{[s,t]}(\tau \wedge T)$. By monotone converge theorem and Remark 4, we derive that $\bar{\mathbb{E}}[\zeta_n - \mathbf{1}_{[s,t]}(\tau \wedge T)] \to 0$, thus $\mathbf{1}_{[s,t]}(\tau \wedge T) \in L_G^1(O^c)$. Then there exists an open set $\tilde{O} \subset O^c$ with $c(\tilde{O}) < \varepsilon$ such that $\mathbf{1}_{[s,t]}(\tau \wedge T)$ is continuous on $O^c \setminus \tilde{O}$. Since $\tilde{O} = A \cap O^c$ for some open subset A in Ω, $O \cup \tilde{O} = O \cup A$ open subset in Ω, and $c(O \cup \tilde{O}) \leq c(O) + c(\tilde{O}) < 2\varepsilon$. Thus $\mathbf{1}_{[s,t]}(\tau \wedge T) \in L_G^1(\Omega)$. $\square$

Proposition 20 *If for any $T > 0$, $\tau \wedge T$ is quasi continuous, then $e^{-r\tau} g(X_\tau)\mathbf{1}_{[\tau<\infty]} \in L_G^1(\Omega)$.*

Proof. For any $T > 0, X_{\tau\wedge T}$ has a quasi continuous version by Proposition 18. Then $e^{-r(\tau\wedge T)} g(X_{\tau\wedge T)})$ has a quasi continuous version. From the variational inequality (8), we have $e^{-r(\tau\wedge T)} g(X_{\tau\wedge T)}) \leq e^{-r(\tau\wedge T)} w(X_{\tau\wedge T)})$. Since $e^{-r(\tau\wedge T)} w(X_{\tau\wedge T)}) \in L_G^1(\Omega)$, $e^{-r(\tau\wedge T)} g(X_{\tau\wedge T)}) \in L_G^1(\Omega)$. By the above lemma, we have proved

$$e^{-r\tau} g(X_\tau)\mathbf{1}_{[\tau\leq T]} = e^{-r(\tau\wedge T)} g(X_{\tau\wedge T})\mathbf{1}_{[0,T]}(\tau \wedge (T+1)) \in L_G^1(\Omega).$$

Notice that $e^{-r\tau}g(X_\tau)\mathbf{1}_{[\tau\le T]} \uparrow e^{-r\tau}g(X_\tau)\mathbf{1}_{[\tau<\infty]}$ $q.s.$ as $T \to \infty$, the proposition follows from monotone convergence theorem. $\square$

Proposition 21 *If the quadratic variation process $\langle X\rangle$ of the diffusion process X is strictly increasing q.s., then for any time $T > 0$ the hitting time $\tau_A := \inf\{t \ge 0; X_t \in A\} \wedge T$ for any Borel set $A \in \mathcal{B}(\mathbb{R})$ has a quasi continuous version.*

Proof. Assume that X is quasi continuous on $[0,T] \times \Omega$. First, let $A = (a,b)$ or $(a,b]$ be an interval ((a,b) can be equal to $(-\infty,\infty), (-\infty,b)$ and (a,∞)). By the same arguments in Theorem 4.5 in Song [S11b], we know that τ_A is a quasi continuous stopping times and is in $L^1_G(\Omega)$.

Let

$$\mathscr{C}_1 = \{(-\infty,a]; a \in \mathbb{R}\}, \quad \mathscr{C}_2 = \{(a,\infty); a \in \mathbb{R}\},$$
$$\mathscr{C}_3 = \{(a,b]; a \le b, a,b \in \mathbb{R}\},$$

then $\mathscr{C} := \mathscr{C}_1 \cup \mathscr{C}_2 \cup \mathscr{C}_3 \cup \{\mathbb{R}\}$ is a semi-algebra. And $\mathscr{C}_{\Sigma f}$ is an algebra.(See pp.4-5 in 'Lecture note on measure theory' by Yan, J.A..)

Set $\mathscr{D} := \{A \in \mathcal{B}(\mathbb{R}); \tau_A \in L^1_G(\Omega)\}$, then $\mathscr{C}_{\Sigma f} \subset \mathscr{D}$. To see this, we consider $A = \sum_{1\le n\le N} A_n, A_n \in \mathscr{C}$. Since $\tau_{A_n} \in L^1_G(\Omega)$ and $\tau_{A_n} \downarrow \tau_A$, we have $\tau_A \in L^1_G(\Omega)$ by monotone convergence theorem and Remark 4.

We can also use monotone convergence theorem to get $\mathscr{D}$ is a monotone class, therefore $\mathscr{D} = \mathcal{B}(\mathbb{R})$, by monotone class theorem. $\square$

Remark 7 *When $\sigma^2(\cdot) > 0$ in the G-SDE* (3)*, X has a strictly increasing quadratic variation process.*

4 Variational Inequality and the Verification Theorem

In the same spirit as the classical optimal stopping theory (e.g., see Shiryaev [S08], Krylov [K80], Bensoussan and Lions [BL82], and Peskir and Shiryaev [PS06]) we expect that the value function v of our optimal stopping problem should identify with an appropriate positive solution w of the variational inequality

$$\max\left\{G(\sigma^2(x)w''(x) + 2\beta(x)w'(x)) + \mu(x)w'(x) - rw(x),\ g(x) - w(x)\right\} = 0, x \in \mathbb{R}, \tag{8}$$

we prove here one version of verification theorem that suits for a classes of optimal stopping problems whose optimal stopping strategy is of a one-threshold type. This is especially suitable when $g(x) = (x-K)^+, g(x) = (K-x)^+$. To this end, we first define the action region $\mathcal{A}$ and the stopping region $\mathcal{S}$ such that

$$\mathcal{A} = \{x > 0 : w(x) > g(x)\}; \tag{9}$$

$$\mathcal{S} = \{x > 0 : w(x) = g(x)\}. \tag{10}$$

We assume that in addition to the above equation (8), the free boundary between $\mathcal{A}$ and $\mathcal{S}$ is $x^* > 0$, such that $w(x)$ is C^2 except at x^* where it is C^1. Moreover, the function w satisfies

$$G(\sigma^2(x)w''(x) + 2\beta(x)w'(x)) + \mu(x)w'(x) - rw(x) = 0 \tag{11}$$

on the set $\mathcal{A}$ and

$$G(\sigma^2(x)w''(x) + 2\beta(x)w'(x)) + \mu(x)w'(x) - rw(x) \leq 0 \tag{12}$$

on the set $\mathcal{S}$.

Theorem 22 *Consider the optimal stopping problem defined by* (1), *and suppose that the variational inequality* (8) *has a solution w as described above and w'' has polynomial growth, i.e. there exist some constant C and an integer $m \in \mathbb{N}$ such that*

$$|w''(x)| \leq C(1 + |x|^m), \quad \textit{for all} \;\; x. \tag{13}$$

Also, consider the stochastic processes Z and R defined by

$$Z_t = e^{-rt}w(X_t) \quad \textit{and} \quad R_t = e^{-rt}g(X_t). \tag{14}$$

The following statements hold true:
(I) *Z is an $(\mathcal{F}_t)$-supermartingale majorising the reward process R, and*

$$v(x) \leq w(x) \quad \textit{for all } x. \tag{15}$$

(II) *We assume that the first hitting time τ_{x^*} of the stopping region $\mathcal{S}$ defined by*

$$\tau_{x^*} = \inf\{t \geq 0 \,|\; X_t \in \mathcal{S}\} \tag{16}$$

is a stopping time in $\mathcal{T}$. If, in addition, w satisfies the uniformly integrable condition: $\{e^{-r(t\wedge\tau_{x^*})}w(X_{t\wedge\tau_{x^*}})\}_{t\geq 0}$ is uniformly integrable. Then τ_{x^*} is optimal, and

$$v(x) = w(x). \tag{17}$$

Proof. w possesses enough regularity for an application of G-Itô formula and yields

$$\begin{aligned} e^{-rt}w(X_t) =& e^{-rs}w(X_s) + \int_s^t e^{-ru}[-rw(X_u) + \mu(X_u)w'(X_u)]\,du \\ &+ \int_s^t e^{-ru}\left[\beta(X_u)w'(X_u) + \frac{1}{2}\sigma^2(X_u)w''(X_u)\right]\,d\langle B\rangle_u \\ &+ \int_s^t e^{-ru}\sigma(X_u)w'(X_u)\,dB_u,\; 0 \leq s < t \leq T. \end{aligned}$$

Noticing that $X\mathbf{1}_{[0,T]} \in M_G^p(0,T)$ for any $p \geq 2$ and $T > 0$, and w, w' have polynomial growth since w'' has polynomial growth, we have

$$\left\{e^{-rt}\sigma(X_t)w'(X_t)\right\}_{0\leq t\leq T} \in M_G^2(0,T).$$

And we have shown before that $w''(X) \in M_G^p(0,T)$ (see Proposition 5), so

$$\left\{ e^{-rt} \left[\beta(X_t)w'(X_t) + \frac{1}{2}\sigma^2(X_t)w''(X_t) \right] \right\}_{0 \le t \le T} \in M_G^1(0,T).$$

Since

$$G(\sigma^2(x)w''(x) + 2\beta(x)w'(x)) + \mu(x)w'(x) - rw(x) \le 0$$

or

$$-rw(x) + \mu(x)w'(x) \le -G(\sigma^2(x)w''(x) + 2\beta(x)w'(x)),$$

we can use Lemma 16 and take conditional G-expectation to derive

$$\begin{aligned}
\hat{\mathbb{E}}_s[Z_t] \le & Z_s + \hat{\mathbb{E}}_s \left[\frac{1}{2} \int_s^t e^{-ru} \eta_u \, d\langle B \rangle_u - \int_s^t e^{-ru} G(\eta_u) \, du + \int_s^t e^{-ru} \sigma(X_u) w'(X_u) \, dB_u \right] \\
= & Z_s,
\end{aligned}$$

where $\eta_u = \left[\sigma^2(x)w''(x) + 2\beta(x)w'(x) \right]_{x=X_u}$. Therefore Z is a G-supermartingale on $[0,T]$ for any $T > 0$.

For $\tau \in \mathcal{T}$, we can derive that $Z_{t\wedge\tau}$ is also a G-supermartingale, just by substituting t and s with $t \wedge \tau$ and $s \wedge \tau$ respectively in the above calculus. Furthermore, equation (15) and the claim that Z majorises R follows immediately from (8) with Z being a positive supermartingale majorising R and from the definition of the value function v and Fatou's lemma for sublinear expectation.

To establish part (II) of the theorem, we observe that, if $\tau_{x^*} \in \mathcal{T}$ is the stopping time defined by (16), then we can apply the G-Itô formula to $e^{-r(t\wedge\tau_{x^*})}w(X_{t\wedge\tau_{x^*}})$ as before and again taking expectations.

$$\begin{aligned}
e^{-r(t\wedge\tau_{x^*})}w(X_{t\wedge\tau_{x^*}}) = & w(x) + \int_0^{t\wedge\tau_{x^*}} e^{-ru}[-rw(X_u) + \mu(X_u)w'(X_u)] \, du \\
& + \int_0^{t\wedge\tau_{x^*}} e^{-ru} \left[\beta(X_u)w'(X_u) + \frac{1}{2}\sigma^2(X_u)w''(X_u) \right] d\langle B \rangle_u \\
& + \int_0^{t\wedge\tau_{x^*}} e^{-ru} \sigma(X_u) w'(X_u) \, dB_u,
\end{aligned}$$

according to the definition of τ_{x^*}, taking G-expectation on both sides of the above equality, we get

$$w(x) = \hat{\mathbb{E}}[e^{-r(t\wedge\tau_{x^*})}w(X_{t\wedge\tau_{x^*}})].$$

By the uniform integrability of $\{e^{-r(t\wedge\tau_{x^*})}w(X_{t\wedge\tau_{x^*}})\}_{t\ge 0}$ and Remark 4, we actually get

$$\hat{\mathbb{E}}[e^{-r\tau_{x^*}}w(X_{\tau_{x^*}})\mathbf{1}_{[\tau_{x^*}<\infty]}] = \lim_{t\to\infty} \hat{\mathbb{E}}[e^{-r(t\wedge\tau_{x^*})}w(X_{t\wedge\tau_{x^*}})\mathbf{1}_{[\tau_{x^*}<\infty]}].$$

Noticing that $g(X_{\tau_{x^*}}) = w(X_{\tau_{x^*}})$ and $e^{-r\tau_{x^*}}g(X_{\tau_{x^*}}) = 0$ on $[\tau_{x^*} = \infty]$, we derive

$$\hat{\mathbb{E}}[e^{-r\tau_{x^*}}g(X_{\tau_{x^*}})] = \hat{\mathbb{E}}[e^{-r\tau_{x^*}}w(X_{\tau_{x^*}})\mathbf{1}_{[\tau_{x^*}<\infty]}] = w(x).$$

Hence $v(x) = w(x)$, and τ_{x^*} is the optimal stopping time in $\mathcal{T}$. ∎

5 Special cases when $v(x)$ is convex

Consider a special G-SDE

$$\begin{cases} dX_t^x = X_t^x(\mu dt + dB_t), \\ X_0^x = x \in \mathbb{R}, \end{cases}$$

then $X_t^x = x\exp\{\mu t - \frac{1}{2}\langle B\rangle_t + B_t\}$.

Lemma 23 (Convexity) *Assume that $g(x)$ is convex, then so is $v(x)$.*

Proof: Since $X_t^x = x\exp\{\mu t - \frac{1}{2}\langle B\rangle_t + B_t\}$, if $g(x)$ is convex, so is $g(X_\tau^x)$. Now, since the supremum of a family of convex functions is still convex, it suffices to show that for any fixed stopping time $\tau \in \mathcal{T}$,

$$V^\tau(x) = \hat{\mathbb{E}}[e^{-r\tau}g(X_\tau^x)]$$

is convex. That is to show that for any $\alpha \in (0,1)$,

$$V^\tau(\alpha x + (1-\alpha)y) \leq \alpha V^\tau(x) + (1-\alpha)V^\tau(y).$$

The above inequality is straightforward from the subadditivity of G-expectation. Indeed, take $z = \alpha x + (1-\alpha)y$, then we have

$$\begin{aligned} & V^\tau(\alpha x + (1-\alpha)y) - \alpha V^\tau(x) - (1-\alpha)V^\tau(y) \\ = & \hat{\mathbb{E}}[g(X_\tau^z)e^{-r\tau}] - \alpha\,\hat{\mathbb{E}}[g(X_\tau^x)e^{-r\tau}] - (1-\alpha)\hat{\mathbb{E}}[g(X_\tau^y)e^{-r\tau}] \\ \leq & \hat{\mathbb{E}}[(g(X_\tau^z) - \alpha g(X_\tau^x) - (1-\alpha)g(X_\tau^y))e^{-r\tau}] \leq 0. \end{aligned}$$

Therefore

$$v(x) = \sup_{\tau\in\mathcal{T}} \hat{\mathbb{E}}[e^{-r\tau}g(X_\tau)]$$

is convex. □

Lemma 24 *If a convex function $w(x)$ is a solution to the nonlinear PDE*

$$\mu\, u_x + G(u_{xx}) = ru$$

where $G(\gamma) = \frac{1}{2}(\overline{\sigma}^2\gamma^+ - \underline{\sigma}^2\gamma^-)$, then w(x) is actually a solution to the linear PDE

$$\mu\, u_x + \frac{1}{2}\overline{\sigma}^2 u_{xx} = ru.$$

Note that if u is convex, then $(u_{xx})^- = 0$, so $G(u_{xx}) = \frac{1}{2}\sigma^2 u_{xx}$.

Proposition 25 *Define $\tau_b = \inf\{t > 0, X_t < b\}, b > 0$. Then*

- $\tau_b \wedge T \in L_G^1$ *for all $T > 0$*

- *Both τ_b and $X(\tau_b)$ are quasi-continuous.*
- $e^{-r\tau_b}(K - X(\tau_b))^+ \in L^1_G$.

Proof. First we know that $\tilde{X}_t := \log(X_t/x) = \mu t - \frac{1}{2}\langle B\rangle_t + B_t$ is a quasi-continuous process (the definition is defined in [S11a]), so is X. Since $\tau_b = \inf\{t > 0, X_t < b\} = \inf\{t > 0, \tilde{X}_t < \log\frac{b}{x}\}$, by Proposition 21, it suffices to show $\langle \tilde{X}\rangle$ is strictly increasing for the first statement. And this is obvious, since $\langle \tilde{X}\rangle = \langle B\rangle$. So we have proved $\tau_b \wedge T \in L^1_G$, and both τ_b and $X(\tau_b)$ are quasi-continuous.

On the other hand, since $\{e^{-r(t\wedge\tau_b)}(K - X(t\wedge\tau_b))^+\}_{t\geq 0}$ is uniformly bounded, it is uniformly integrable. Hence we have

$$\bar{\mathbb{E}}[|e^{-r(t\wedge\tau_b)}(K - X(t\wedge\tau_b))^+ - e^{-r\tau_b}(K - X(\tau_b))^+|] \to 0.$$

Noticing that $e^{-r(t\wedge\tau_b)}(K - X(t\wedge\tau_b))^+ \in L^1_G(\Omega)$ for each $t > 0$, by Remark 4 we have $e^{-r\tau_b}(K - X(\tau_b))^+ \in L^1_G$. □

Lemma 26

$$\hat{\mathbb{E}}[e^{-r\tau_b}g(X((\tau_b))] = E^{P^{\overline{\sigma}}}[e^{-r\tau_b}g(X((\tau_b))],$$

where $P^{\overline{\sigma}} \in \mathcal{P}_M$ is define by $P^{\overline{\sigma}} = P^0 \circ a^{-1}, a_t := \int_0^t \overline{\sigma}dW_t = \overline{\sigma}W_t$.

Proof. Since the value function is convex, the related nonlinear PDE is actually a linear PDE

$$\mu x v_x + G(x^2 v_{xx}) - rv = \mu x v_x + \frac{1}{2}\overline{\sigma}x^2 v_{xx} - rv = 0.$$

And under probability $P^{\overline{\sigma}}$, $\langle B\rangle_t = \overline{\sigma}^2 t\, a.s.$, thus by Lévy's martingale characterization of Brownian motion we see that $\tilde{B}_\cdot := B_\cdot/\overline{\sigma}$ is a Brownian motion.

6 Appendix

Proof of Proposition 5 Let $(X^{(m)})$ be the Picard iterative approximation sequence defined by

$$X^{(0)} \equiv X_0,$$
$$X^{(m)} = X_0 + \int_0^t \mu(X_u^{(m-1)})\,du + \int_0^t \beta(X_u^{(m-1)})\,d\langle B\rangle_u + \int_0^t \sigma(X_u^{(m-1)})\,dB_u, \quad m \geq 1.$$

We claim that $X^{(m)}\mathbf{1}_{[0,T]} \in \bar{M}^p_G(0,T)$ for any $p \geq 2, m \in \mathbb{N}$.

First, we have $X^{(0)} \in \bar{M}^p_G(0,T)$ and $X^{(1)} \in \bar{M}^p_G(0,T)$. Second, for $m \geq 1$, by Hölder inequality,

we have

$$
\begin{aligned}
|X_t^{(m+1)} - X_t^{(m)}|^p =& \bigg| \int_0^t [\mu(X_u^{(m)}) - \mu(X_u^{(m-1)})]\, du + \int_0^t [\beta(X_u^{(m)}) - \beta(X_u^{(m-1)})]\, d\langle B\rangle_u \\
&+ \int_0^t [\sigma(X_u^{(m-1)}) + \sigma(X_u^{(m-1)})]\, dB_u \bigg|^p \\
\leq& 3^{p-1}\bigg\{ \bigg| \int_0^t [\mu(X_u^{(m)}) - \mu(X_u^{(m-1)})]\, du \bigg|^p + \bigg| \int_0^t [\beta(X_u^{(m)}) - \beta(X_u^{(m-1)})]\, d\langle B\rangle_u \bigg|^p \\
&+ \bigg| \int_0^t [\sigma(X_u^{(m-1)}) + \sigma(X_u^{(m-1)})]\, dB_u \bigg|^p \bigg\} \\
\leq& 3^{p-1}\bigg\{ T^{p-1} \int_0^t \Big| \mu(X_u^{(m)}) - \mu(X_u^{(m-1)}) \Big|^p du \\
&+ (\overline{\sigma}^2 T)^{p-1} \int_0^t \Big| \beta(X_u^{(m)}) - \beta(X_u^{(m-1)}) \Big|^p d\langle B\rangle_u \\
&+ \bigg| \int_0^t [\sigma(X_u^{(m-1)}) + \sigma(X_u^{(m-1)})]\, dB_u \bigg|^p \bigg\}.
\end{aligned}
$$

Then by Theorem 2.1 in Gao [G09], we get

$$
\begin{aligned}
&\bar{\mathbb{E}}[\sup_{0\leq t\leq T} |X_t^{(m+1)} - X_t^{(m)}|^p] \\
\leq& 3^{p-1} T^{p-1} L^p (1+\overline{\sigma}^{2p}) \int_0^T \bar{\mathbb{E}}[|X_u^{(m)} - X_u^{(m-1)}|^p]\, du \\
&+ 3^{p-1}\Lambda_p \overline{\sigma}^p L^p \bar{\mathbb{E}}\left[\left(\int_0^T |X_u^{(m)} - X_u^{(m-1)}|^2\, du \right)^{\frac{p}{2}} \right] \quad \leq \quad C_{p,T} \int_0^T \bar{\mathbb{E}}[|X_u^{(m+1)} - X_u^{(m)}|^p]\, du,
\end{aligned}
$$

where Λ_p is a constant depend only on p, and $C_{p,T} = 3^{p-1}L^p[T^{p-1}(1+\overline{\sigma}^{2p}) + \Lambda_p \overline{\sigma}^p T^{\frac{p}{2}-1}]$.

Thus, if $X^{(m-1)}, X^{(m)} \in \bar{M}_G^p(0,T)$, then $X^{(m+1)} \in \bar{M}_G^p(0,T)$. By induction, we have proved the claim. Since $\phi = \mu, \beta, \sigma$ satisfy conditions (I)(II), $\phi(X^{(m)})\mathbf{1}_{[0,T]} \in \bar{M}_G^p(0,T)$ for each $m \geq 1$, and $X^{(m)}$ is well-defined.

Set $\Delta_m(t) = \bar{\mathbb{E}}[\sup_{0\leq s\leq t} |X_s^{(m)} - X_s^{(m-1)}|^p]$. Then we have proved $\Delta_1(T) < \infty$ and for any $t \in [0,T]$,

$$
\Delta_m(t) \leq C_{p,T} \int_0^t \Delta_{m-1}(u)\, du.
$$

Therefore,

$$
\Delta_m(T) \leq \Delta_1(T)(C_{p,T})^m \frac{T^m}{m!}
$$

and so

$$
\sum_{m=1}^{\infty} \bar{\mathbb{E}}[\sup_{0\leq s\leq T} |X_s^{(m)} - X_s^{(m-1)}|^p] < \infty
$$

which yields

$$c\left(\sum_{m=1}^{\infty}\sup_{0\le s\le T}|X_s^{(m)}-X_s^{(m-1)}|=\infty\right)=0.$$

Set $\tilde{\Omega}=\cap_{k=1}^{\infty}\{\sup_{0\le s\le k}|X_s^{(m)}-X_s^{(m-1)}|<\infty\}$ and $X_t=\sum_{m=1}^{\infty}(X_t^{(m)}-X_t^{(m-1)})+X_0=\lim_{m\to\infty}X_t^{(m)}$. Then $c(\tilde{\Omega}^c)=0$, and for any $\omega\in\tilde{\Omega}$, X is continuous. Moreover,

$$\bar{\mathbb{E}}[\sup_{0\le s\le T}|X_s^{(m)}-X_s|^p]\to 0 \text{ as } m\to\infty.$$

On the other hand, by

$$\bar{\mathbb{E}}[\sup_{0\le s\le T}|X_s^{(m)}-X_s^{(m+l)}|^p]\to 0$$

for any $l\ge 1$ as $m\to\infty$, we see that $\{X_t^{(m)},m\ge 1\}$ is a Cauchy sequence in $L_G^p(\Omega_t)$, and $X_t\in L_G^p(\Omega_t)$.

Now by the Lipschitz continuity and Theorem 2.1 in Gao [G09], there exists a constant $C>0$ such that

$$\bar{\mathbb{E}}\left[\sup_{0\le t\le T}\left|\int_0^t\mu(X_u^{(m)})\,du-\int_0^t\mu(X_u)\,du\right|^2\right]\le C\int_0^T\bar{\mathbb{E}}[|X_u^{(m)}-X_u|^2]\,du,$$

$$\bar{\mathbb{E}}\left[\sup_{0\le t\le T}\left|\int_0^t\beta(X_u^{(m)})\,d\langle B\rangle_u-\int_0^t\beta(X_u)\,d\langle B\rangle_u\right|^2\right]\le C\int_0^T\bar{\mathbb{E}}[|X_u^{(m)}-X_u|^2]\,du,$$

and

$$\bar{\mathbb{E}}\left[\sup_{0\le t\le T}\left|\int_0^t\sigma(X_u^{(m)})\,dB_u-\int_0^t\sigma(X_u)\,dB_u\right|^2\right]\le C\int_0^T\bar{\mathbb{E}}[|X_u^{(m)}-X_u|^2]\,du,$$

hence, X is a solution to the G-SDE (3).

Next let us prove the uniqueness. Let X and Y be two solutions to (3). Set

$$\Delta(t)=\bar{\mathbb{E}}[\sup_{0\le s\le t}|X_s-Y_s|^2].$$

Then as in the proof of existence, we can derive

$$\Delta(t)\le C(T)\int_0^t\Delta(u)\,du,$$

for some constant $C(T)$. Thus, by Gronwall's inequality, $\Delta(t)=0$ which follows $X=Y$ *q.s.*. ∎

Remark 8 *From the above proof, we have the following estimation*

$$\bar{\mathbb{E}}[\max_{0\le s\le t}|X_s|]\le\sqrt{\Delta_1(t)}e^{C_{2,t}t/2}+|x|.$$

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